Forming the concept of parameter through the work with computer algebra software

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Abstract

The aim of the study is to investigate whether software can be useful in a process of forming the concept of parameter. The author also would like to show to what extent computer program can help in understanding different meanings of this concept during continuous contact with this tool in mathematics learning. Pupils at the age of 13 – 16 used different computer programs during a three-year period. They studied the parameter in different roles: placeholder, generalizer, changing quantity and unknown-to-be-found. The results indicate that the integration of mathematics and information technology could contribute pupils' understanding of algebra concepts, particularly the concept of parameter.

1. Introduction

Many researchers investigated opportunities of integration technology into mathematics education. Some of them were interested in answering the question how the use of computer algebra can contribute to understanding of mathematical concepts (Drijvers, 2003; Giessen van de, 2002). These researchers focused on the concept of parameter. They concluded that the work in the computer algebra environment supported the understanding of different meanings of this concept. They also conjectured that the contribution of computer algebra to the conceptual development would improve if it were used for a longer period, so that instrumentation difficulties were less dominant. The appearance of computer algebra programs in Polish schools has provoked a continuous research on process of forming mathematical concepts with the use of that tool, in order to check if the use of computer algebra can promote algebraic understanding. This knowledge is essential before computer programs are included into this process. The study of references shows a deficiency of research in that field. Another thing - the use of a computer recording program during research enabled a deeper insight into the process of forming the concept of parameter with the use of the computer programs.

2. Background

Computer algebra systems seem to be ideal technologies for fostering growth in the level of understanding of mathematical concepts. Using the CAS, students can signify (and execute) actions on mathematical entities without needing to carry out the procedures by hand (Heid, 2003). CAS assists with routines but does not take over the role of mathematical thinking. (Pierce and Stacey, 2001). The most common object of interest in traditional mathematics teaching are contents connected with concepts like triangles, numbers, vectors etc. and their properties. A teacher focuses pupils' attention mainly on these concepts. Letters (variables) are treated as if they were already known to pupils and represented the part of alphabet earlier known to them. It is clear that the assumption that pupils will use mathematical symbols in a conscious manner is wrong as long as nobody has familiarized them with these symbols (Konior, 2002). A context determines their meaning in the algebraic text, phase of reasoning, subjective interpretation and also the customs concerning the kind of used letters. A proper interpretation of letters in a given algebraic structure is often a task overgrowing pupils. Feeling and intuitive recognition turns out to be more popular strategy than interpreting of a letter in the way of a certain reasoning. Pupils know concepts like a variable, an unknown or a parameter. However, they do not always use them consciously, and therefore they are not understood by them entirely (Turnau, 1990). One of researchers described how the meanings attached to literal symbols can shift during the process of solving problems (Bills, 2001). Another classified the different roles a letter can have as an unknown, a general number, a variable and a parameter (Usiskin, 1988). The study presented here is focused on the letter in the meaning of parameter. There is no definition of the concept of parameter in mathematical literature. According to some of descriptions of that concept *a parameter can be considered as a meta-variable: the a in* $y = a \cdot x + b$ *can play the same roles as an 'ordinary' variable, such as placeholder, unknown or changing quantity, but it acts on a higher level than is the case for a variable* (Drijvers, 2001). Another more formal description says that *an equation or a function with a parameter is a (second order) function, the argument of which is the parameter, and the corresponding values of which are equations and functions (with the other letters as unknowns and variables)* (Bloedy-Vinner, 2001). Only a few studies paid attention to students' work with parameters (Bloedy-Vinner, 1994, Bloedy-Vinner, 2001, Furinghetti & Paula, 1994). Some of them concluded that the majority of the students they surveyed, had difficulty in expressing the difference between an unknown, a variable and a parameter (Furinghetti & Paula, 1994). Another researchers stated that the interpretation, manipulation and symbolization of parameters was, in general, difficult for students (Ursini & Trigueros, 2004).

Another research involved the concept of parameter with using information technology. One of the researchers described classroom experiences of 16-17 year-old students connected with visualization of different meanings of the concept of parameter during solving problems with the use of program Graphic Calculus (Giessen van de, 2002). Besides, in 2003 at the University of Utrecht in Holland, the research was performed on the understanding of the concept of parameter with using computer programs (Drijvers, 2003). In those two experiments four meanings of the concept of parameter were classified: placeholder, generalizer, changing quantity and unknown-to-be-found. The following table presents these meanings with their descriptions:

Table 1: *Meanings of a parameter*

Meaning of a parameter	**Algebraic meaning**	**Graphic image**
placeholder	-one numerical value, -one at a time	-one graph, -one at a time, -a parameter in this role has a static character
generalizer „family" parameter	a set of numerical values	a sheaf of graphs
changing quantity	a changing numerical	-a graph is changing continuously with

„sliding” parameter	value	change of a parameter, - a parameter has a dynamic character
unknown-to-be-found	a subset of numerical values	a subset of graphs in a sheaf

3. Aims and methods

My case studies aim at answering the following question:

- Can software be useful in the process of forming the concept of parameter?

This question needs further specification.

- To what extent can computer algebra software help student in understanding different meanings of the concept of parameter during a continuous contact with this tool in mathematics learning?

The above mentioned second question asks whether computer algebra software can help to achieve the extensions of the understanding of the parameter from placeholder to the parameter as changing quantity, as generalizer and as unknown.

The study had a continuous character and was conducted during a three-year period at junior high school (13-16 year-old pupils). The subjects of the study were pupils of two classes of this school (Jan Paweł II Junior High School in Kęty) who were formed not in a selective way. I was the teacher in those classes and also the researcher. In the oldest class D - I carried three-year preliminary research and in the youngest one B proper research which also lasted three years. The figure below shows this situation:

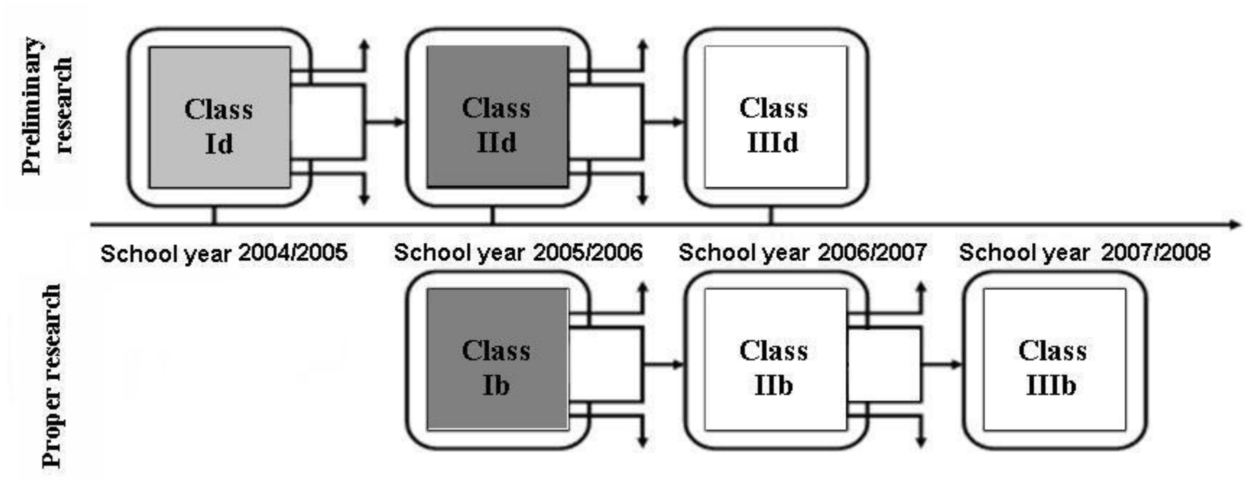

Figure 1: *Schedule of preliminary and proper research*

During preliminary research there were created, developed and verified lists of problems connected with the parameter for the first, the second and the third class of high school, each related to other section of mathematics. Afterwards these lists were destined for the proper research. After some time of carrying this research problems with parameters did not seem a new type of problems to the pupils.

As a teacher of mathematics and information technology I decided to use software for solving different mathematical problems during lessons of mathematics in order to contribute to the understanding of the concept of parameter by pupils. I led lessons of mathematics in a traditional manner, as well as with the use of software. Pupils were accustomed to solving various mathematical problems with the use of software during these lessons. I also decided to use computer programs in the research described here. The reason of this choice could be one of final conclusions of certain research: *a computer can help with the understanding of mathematical concepts taking a stand in a solved problem, illustrating, for example, images of these concepts* (Kąkol & Ratusiński, 2004). Such observation was revealed in this research in connection with the concept of parameter.

During the lessons of information technology pupils were acquainted with operating techniques of different software, for example TI InterActive**!.** Abilities of using this program played role at solving mathematical problems only in the initial phase of experiment. Pupils did not use any sheets of paper to solve the problems with the aid of this program, because this program could be used as a text editor, and it was possible to insert objects, for example SliderContol (for changing values), object Graph (for composing graph of function), object MathBox (for symbolic accounts). At the beginning of work with this program researched children decided which objects were important in the problem and in which moment of solving the problem use it. They only had the program available and created these objects on their own. The files of program TI InterActive! including text and objects, were recorded by them on the disk of their computers. I analyzed the solutions of the problems with the parameter from the lessons and the final tests during which pupils also could use software.

The most important data consisted of the files of the computer programs which were created independently by pupils while solving problems. However, the most valuable investigative tool was a screen recording software Camtasia Studio which made it possible to record the work of each of the examined pupils. It is essential that the recording program allows to replay pupil's work in the computer program. Therefore, the files of computer programs and the recordings of pupils' work were used for the analysis of the solutions of the problems. In order to achieve a deeper insight into that way of understanding of the concept, it was crucial to supplement the recording with a interview with some pupils about their problem solving capabilities.

This research was consistent with the idea of natural experiment. *Design research* method supported by classroom observations and analysis of the documents also was used in this investigation. According to this method, the process of the research has a cyclic character with macro- and microcycles. One macrocycle lasted one year and included the phases: planning, conducting of the experiment and analysis of the results of the research involving comparison with the initial state. In turn, one microcycle concentrated on details containing one section of mathematics. Planning of the research within the confines of one microcycle consisted of the preparation of the list of problems that refer to different meanings of the concept of parameter.

4. Results within the right triangle problem

The application of the computer recording program enabled a thorough investigation and an exact reconstruction of the pupil's work when solving a problem with the use of a computer program. In this section I present the analysis of a solution proposed to a problem made by a girl of class II B. At the beginning of the research she was one of the two girls who understood the parameter as a finite generalizer. Let me concentrate on the work of this girl.

What are the values of the parameter n for which points $A=(2,1)$, $B=(-3,2)$, $C=(2n-1,1-n)$ are the vertices of a right triangle at a right angle at A vertex.

a) The description of student's work

The girl used TI InterActive! program, and before solving this problem, she created a tool called SliderControl. She established a letter n meaning the parameter for such a tool. Initially, in this program, the current value of this parameter is equal to 0 and the value of the step of assigning consecutive values the parameter equal to 0,1. This pupil did not change these settings at that moment. She also inserted a tool Graph, and then, after choosing a Stat Plots tab, she typed the coordinates of given three points which were then plotted in the coordinate plane. The first parameter value tried by this pupil was an initial value. At that moment she decided to change the parameter value in order to check which point will move. She assessed that it was point C.

Next, she added labels to these points using letters A, B. She did not add a label for point C, because this point was changing positions as parameter values changed, but the label did not. Next, she tried to find parameter values which would allow a triangle to be rectangular at a right angle at A vertex. For this purpose, she changed the values of this parameter by using the slider, conducted many these trials, and observed how the position of point C changed in relation to points A and B. She found the parameter value at which a triangle was rectangular at a right angle at A vertex. The pupil states a hypothesis that this value amounted to 1,3 (Figure 2).

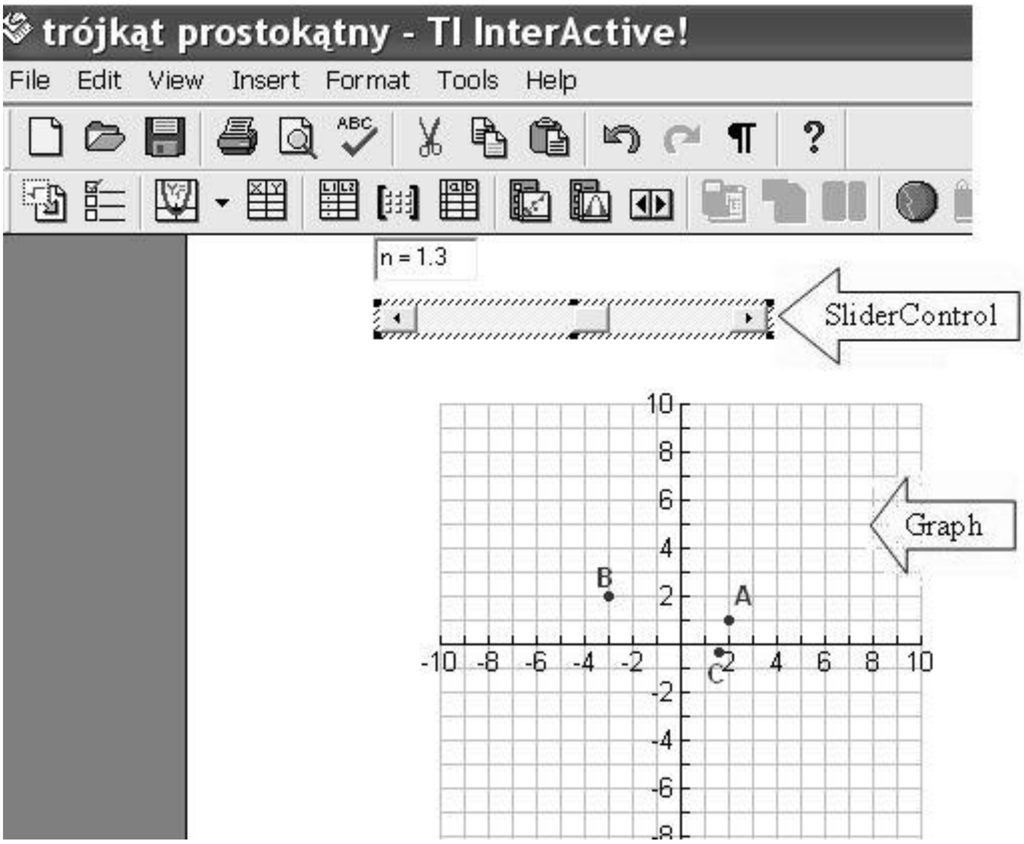

Figure 2: *The file of TI InterActive! program with the parameter value amount of 1,3*

After the initial estimation of the parameter value she approached the calculations. Using the Math Box tool, enabling executing symbolic accounts, she calculated the distances between these points. She marked the distance between points A and B as a and calculated its value typing $a=\sqrt{(-3-2)^2+(2-1)^2}$. This distance was not dependent on parameter n. Then she marked the distances between points B and C as b and between points A and C as c. Then it was calculated that b amounted to $\sqrt{(2n-1-2)^2+(1-n-1)^2}$ and c amounted to $\sqrt{(2n-1+3)^2+(1-n-2)^2}$. In this way she found the formulas for these distances expressed in the parameter n. Next she wanted to know numerical values of the expressions a^2+c^2 and b^2. Using the Math Box tool, she calculated their numeric values. However, for the initial marked value of the parameter n equal to 1,3, the expressions had almost the same numeric values. She then thought that probably the triangle would be rectangular when they would be equal. Since these values differed insignificantly, she activated SliderControl tool and reduced the step value from 0,1 to 0,01. Then using the SliderControl tool she changed the current value of the parameter. She conducted six trials and ascertained that the above-mentioned expressions had approximate values for parameter n of 1,36 (Figure 3).

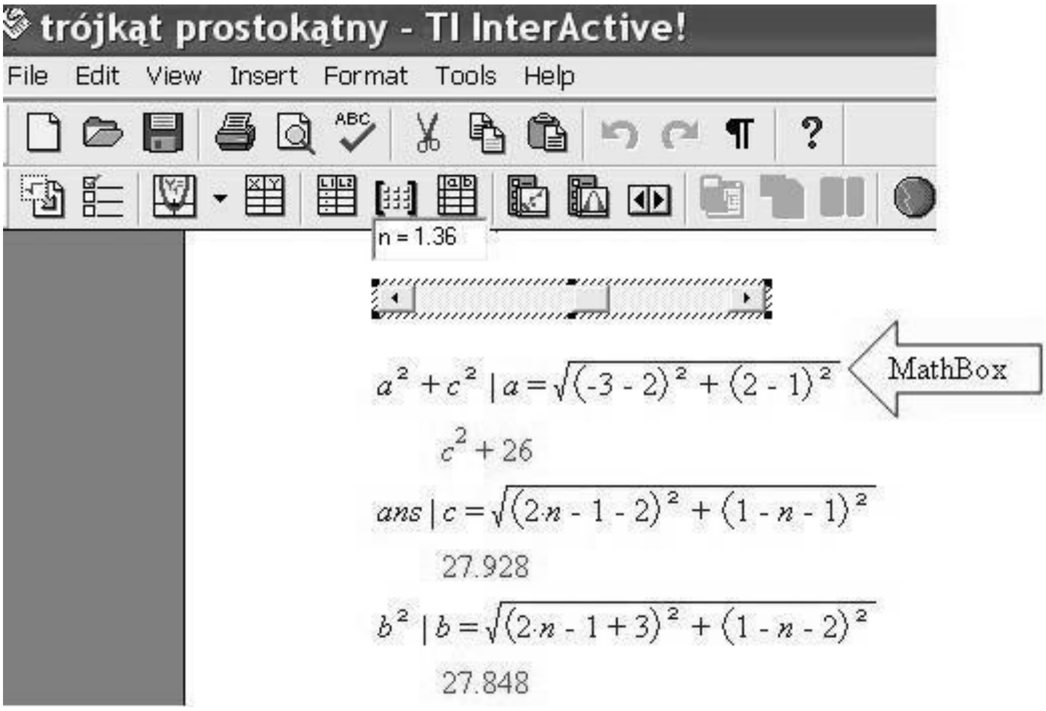

Figure 3: *The file of TI InterActive! program with the parameter value amount of 1,36*

She carried out another value of the step of assigning consecutive values of the parameter from 0,01 to 0,001. Then, using the SliderControl tool she changed the current value of parameter. She conducted three trials and she showed that the expressions had approximate values for the parameter n equal to 1,363. After a following decrease of the step value from 0,001 to 0,0001 she conducted six trials and claimed that these expressions had approximate values for the parameter n equal to 1,3636. After a reduction of the step value to 0,00001 she conducted five trials and found the parameter n value of 1,36363. Next this girl reduced the step value to 0,000001. She conducted fifteen trials and found that numeric values of the expressions were equal for the parameter value equal to 1,363635; 1,363636; 1,363637 and 1,363638. Some of these trials concerned the same parameter values, I mean she checked the parameter value 1,363635 a few times. Finally, she stated a hypothesis that the exact value of the parameter n probably would be 1,(36). She verified the hypothesis by entering in the MathBox tool the following command:

$$solve((-3-2)^2+(2-1)^2+(2n-1-2)^2+(1-n-1)^2=(2n-1+3)^2+(1-n-2)^2,n).$$

She received a result $n=\frac{15}{11}$ that meant infinite decimal fraction 1,(36). Therefore, she confirmed the solution which was solved earlier using the SliderControl tool.

However the problem solving did not desist at this moment, rather she decided to extend the problem. She asked herself an additional question about the cases when the value of the parameter was different from 1,(36). First she considered values of the parameter n that the point C is on the right of points A and B. In this situation the longest side of the triangle ABC is the side between points B and C; with this knowledge, she compared the numerical values of the expressions a^2+c^2 and b^2. She realized that the numerical value of a^2+c^2 is less than that of b^2 so this triangle was an obtuse triangle. Next she considered values of the parameter n that point C is on the left of points A and B. In this circumstance the longest side of the triangle ABC is the side between points A and C. Thus, calculating the numerical values of the expressions a^2+b^2 and c^2 she saw that the numerical value of a^2+b^2 is less than the numerical value of c^2 so this triangle was an obtuse triangle. Then she considered values of the parameter n that point C is between points A and B. For the majority of parameter values the longest side of the triangle ABC is the side between points A and B, so she calculated numerical values of the expressions c^2+b^2 and a^2. She concluded that the numerical value of c^2+b^2 is less than the numerical value of a^2 therefore, this triangle was an obtuse triangle. In the situation where point C is between points A and B for some values of the parameter, the longest side of the triangle is the side between points C and B. She calculated the numerical values of the expressions a^2+c^2 and b^2 and realized that numerical value of a^2+c^2 is less than the numerical value of b^2 making this triangle obtuse.

b) Analysis

An analysis of the collected material allows an opportunity to make some remarks about this girl's mathematical abilities revealed while solving the problem:

- *Detecting analogies*: By substituting different values of the parameter student repeated problem solving procedures for these particular values. Then she detected analogies between these procedures.
- *Generalization*: Solving the problem with the parameter in the computer program allowed for varying parameter values and calculating generic algebraic solutions (Drijvers, 2003). As noticed in the above analysis, solving the problem with the parameter also fosters generalizing through assigning different parameter values and observing situations connected with those values.
- *Stating and verifying hypotheses*: The results were not regarded as a final solution of the problem. They were treated as a hypothesis which could be verified by calculations.
- *Use of symbolic language*: She calculated the distances between points in general. Her solving strategy did not seem to be confused by the presence of the parameter.

c) Software can be useful in the process of forming the concept of parameter, because it allows for:

- *Experimentation*: With the CAS, students can generate and manipulate symbolic expressions that were otherwise too time-consuming, too effortful or too complicated for students to handle in the course of their problem solving (Heid, 2003). Due to the use of the SliderControl tool for solving problems with help of TI InterActive! program she could substitute different parameter values. Any number could be substituted for the parameter, not only integer (not only positive as well), as usually it is accepted during solving problems without use of computer programs. This girl observed the situation given in the problem at different parameter values. Next she formulated hypotheses and verified them.
- *Visualization of step by step changes of parameter value*: Changes of parameter value were made step by step and the consequences of these changes could not be as clearly noticed when solving this problem using „paper and pencil". Such observations could be imagined only if the changes had to concern a finite set of values. However it could also involve an infinite set of values, e.g. with the value of the step of changing parameter values equal to 0,1.
- *Visualization of continuous changes of parameter value*: CAS affords the user the opportunity to transcend the limitations of the mind, through its capacity to generate a larger number and a greater range of "examples" for students to encounter (Heid, 2003). Using the SliderControl tool while solving the problem helped her to understand that the parameter is a kind of generalization and it substitutes any number; that means it is possible to insert any numerical value in its place. Due to this tool she observed that the parameter accepts certain values one by one, and one value at any time. Understanding that parameter values dynamically run through a set of values would be harder without the aid of software (Giessen van de, 2002).
- *Animation of situation given in problem*: Due to the allowance of animation in this program, she was able to notice that each concrete value of parameter is attributed to one numerical value of the expression. In other words, one particular case of the problem is assigned for this value. She saw that along with the change of parameter value, ensuing consequences in the coordinates or the expressions' of distances. This fact would be hard to show without the use of software. The use of the slider bar illustrated the parameter as changing quantity and the link between graphical features and algebraic properties (Drijvers, 2003).
- *Achieving symbolic accounts*: In TI InterActive! program it is possible to solve problems with parameter without an assignment of any particular value. For example, while solving an equation step by step, it is possible to treat parameter as a given established value. However, she has rarely benefited from such an opportunity of solving problem. It was easier for her to establish one concrete value of parameter and then analyze the solution of the problem.
- *Focusing attention on given problem*: The girl wrote several sentences about her work with TI InterActive! program during mathematics lessons. *"Program TI InterActive! is very beneficial. It can be used as a computer exercise book. Working with it is very good and easy. It is not necessary to concentrate on counting, but only on correct writing operations.*" It happens relatively often that pupils are lost while solving a problem with parameters. They do not remember the aim and that is why software is taking over the computational part of the problem allowing the pupils to concentrate on the essence of problem.

5. Conclusions

An analysis of collected material enables me to formulate partial answers to previously asked questions.
Computer program can help in the understanding different meanings of the concept of parameter:

- The pupil deepened her understanding of the concept of parameter as generalizer, because she remarked that the parameter can accept an infinite number of values according to established step value. While working with software she could perceive how values of the parameter were varying in jumps according to established value of the step of assigning consecutive values of parameter equal to 0,1.
- This girl perceived, along with change of parameter values, dynamic changes of the problem in the coordinate plane. In order to get the exact value of the parameter she had to reduce the step value. It demonstrated change of parameter value in a continuous manner. The solution of the problem indicates how her understanding of the concept of parameter developed from the finite generalizer to changing quantity.
- Furthermore, the pupil used *the solve* command. She solved the equation with respect to parameter n. In this situation her understanding of the concept of parameter developed from the finite generalizer to unknown-to-be-found.

This girl was very good in mathematics, had a good attitude and willing to talk about her work. The above example of her work shows that computer algebra software not only helps in understanding different meanings of the concept of parameter during a continuous contact with this tool in mathematics learning. It also makes it possible to achieve the extensions of the understanding of the parameter from placeholder to the parameter as changing quantity, as generalizer and as unknown. However, computer algebra software was a difficult tool for students of the age and level of the subjects of this research. I conjecture that integrating CAS into algebra education would be better suited in higher grades, when pupils have more algebraic experience.

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